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Matchings, cutsets, and chain partitions in graded posets

Jerrold R. Griggs*

Center for Communications Research, 4320 Westerra Court, San Diego, CA 92121 USA

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Abstract

New properties that involve matchings, cutsets, or skipless chain partitions in graded posets are introduced and compared to familiar Sperner and chain partition properties. Related work is surveyed. We determine all possible combinations of these properties, with the exception of a long-standing open conjecture about *LYM* posets, and provide a list of examples realizing these combinations.

1. Introduction

In an earlier paper [12], we considered the connections between the Spencer property (and its generalizations) and chain partitions in graded posets. Here we collect the most important of these properties and compare them to new ones involving matchings between pairs of ranks, minimum cutsets, or skipless chain partitions. Implications among the properties are determined for the class of graded posets, with the exception of one long-standing conjecture about *LYM* posets. Examples of posets realizing the possible combinations of properties are presented. The construction of these posets was technically harder than obtaining the theorems. These posets may prove valuable for testing future conjectures. To assist the reader, a list of all properties is included in the appendix.

In the next section, we review the required basic poset terminology. The notion of a k-cutset, while quite natural, appears to be new in the poset setting. The following section reviews results linking Sperner and chain decomposition properties. A new chain condition, called the skipless Dilworth property D, is introduced.

We introduce properties M_1 and M, which concern the existence of matchings between levels in graded posets, in Section 4. Theorem 4, our principal positive result,

* E-mail address: griggs@math.scarolina.edu. Permanent address: Department of Math., University of South Carolina, Columbia, SC 29208, USA. Research supported in part by NSA/MSP Grants MDA 90-H4028 and 904-92H3053.

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relates the matching property M to the Sperner and chain properties. Section 5 introduces properties C_1 and C concerning the minimum size of cutsets and k-cutsets to the other properties. We review the current status of LYM (normalized matching) posets in Section 6.

In Section 7 we summarize our findings. The possible implications are described in Figs. 1–3 for the class of graded posets in general and for the classes of graded posets that are rank-unimodal or both rank-unimodal and rank-symmetric. Poset examples in Fig. 4 realize the combinations of properties. We also discuss the extension of these results to ranked posets in general.

2. Terminology

We consider finite posets (ordered sets) P. Let $C = \{x_1 < \cdots < x_r\}$ be a chain in P. The *length* of C is $r - 1 = |C| - 1$. We say chain C is *skipless* if x_i covers x_{i-1} for all i. (A referee suggested trying 'skipless' or 'internally-maximal' in place of the usual term, 'saturated', which one can confuse with the notion of a saturated chain partition.)

We say P is *graded* if every maximal chain in P has the same length. A *ranked* poset is a poset P together with a function $r: P \to \{0, 1, \ldots\}$ such that $r(y) = r(x) + 1$ whenever y covers x in P. A graded poset is ranked since for every $x \in P$, every skipless chain from a minimal element up to x has the same length, which we take as $r(x)$, the *rank* of x. For example, the following subsets ordered by inclusion form a ranked, but not graded, poset: $P = \{\{a\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,b,e\}, \{a,b,c,d\}\})$ is ranked by $r(X) = |X|$. Unless stated otherwise, posets in this paper are assumed to be graded. For a chain C in such a poset, $r(C)$ denotes the set of ranks of elements in C.

Suppose P is ranked with minimum rank zero. We say P has *rank* $r(P) := \max r(x)$. Let P_i denote the ith *rank set* consisting of elements of rank i in P. The (infinite) *sequence of Whitney numbers* of P is $(|P_0|, |P_1|, |P_2|, \ldots)$. We say P is *rank-unimodal* (property RU) if there exists an I such that

$$|P_0| \leqslant \cdots \leqslant |P_{I-1}| \leqslant |P_I| \geqslant |P_{I+1}| \geqslant \cdots .$$

We say P is *rank-symmetric* if for all i, $|P_i| = |P_{n-i}|$, where $n = r(P)$.

Many important poset families are rank-symmetric and rank-unimodal (property RSU), including the Boolean lattice B_n of subsets of an n-set, the linear lattice $L_n(q)$ of subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$, and the lattice $L(m,n)$ of order ideals of a product of an m-chain and an n-chain. The lattice Π_n of partitions of $\{1, \ldots, n\}$, ordered by refinement and ranked by $r(x) = n - k$, where k is the number of blocks of x, is rank-unimodal, but not rank-symmetric.

It is also helpful to order the Whitney numbers by size. Let M_j be the $(j+1)$st largest $|P_i|$ for $j \geqslant 0$.

The *width* $d_1(P)$ of any poset P is the maximum size of an *antichain* (totally unordered subset, also known as a Sperner family). More generally, for any $k \geqslant 1$, let

$d_k(P)$ denote the maximum number of elements in any k-family F, which is a subset of P that contains no chain of size $k + 1$. One may verify by removing the set of maximal elements of F and applying induction on k that a k-family is any union of at most k antichains.

For any rank sets A and B of a ranked poset P, we say there is a *matching* from A to B, written $A \hookrightarrow B$, if there exists an injection $f: A \to B$, such that for all $a \in A$, $f(a)$ is related to a in P. We say the *shadow of* $X \subseteq A$ *in* B, denoted $\Gamma(X)$, consists of all elements in B related to some element in X.

A *cutset* in any P is a subset that meets (*i.e.*, cuts) every maximal chain. More generally, for any $k \geqslant 1$, we say $F \subseteq P$ is a *k-cutset* if $|F \cap C| \geqslant k$ for every maximal chain $C \subseteq P$. Let g_k denote the minimum size of any k-cutset of P, when one exists.

We define two poset operations. The *disjoint union* of posets P and Q is denoted by $P + Q$. The *ordinal sum* $P \oplus Q$ places Q over P: It has elements $P \cup Q$ ordered by $x \leqslant y$ when $x \in P$ and $y \in Q$, or when $x, y \in P$ (or $x, y \in Q$) with $x \leqslant y$ in P (or in Q).

We let A_k denote an antichain of size k and let C_k denote a chain of size k (length $k - 1$).

3. Sperner and chain conditions

There is a duality between antichains and chain conditions that is fundamentally demonstrated by Dilworth's theorem [6] that every finite poset P has a partition into just $d_1(P)$ chains. In graded posets, each rank set is an antichain, so $d_1(P) \geqslant \max_i |P_i| = M_0(P)$. Sperner's theorem [19], a fundamental 1928 discovery, is that when P is the Boolean lattice B_n, the width is just $\binom{n}{\lfloor n/2 \rfloor}$, which is $d_1(B_n)$. Rota [18] asked in the 1960s whether the partition lattice Π_n has the same property that $d_1(P) = M_0(P)$, now called the *Sperner property* S_1. A graded poset with property S_1 has a partition into just $M_0(P)$ chains. Although Rota's conjecture for Π_n was verified for small n, it was finally disproved by Canfield [4] in 1978, and shown to be false for all (very) large n.

For any $k \geqslant 1$ it is true more generally that any union of k rank sets in a graded poset is a k-family, so in particular, $d_k(P) \geqslant M_0 + M_1 + \cdots + M_{k-1}$. When equality holds for k, we say that P is *k-Sperner*. We say P has the *strong Sperner property S* if it is k-Sperner for all k [12]. Obviously, property S implies S_1, but Sperner posets are not in general strongly Sperner posets. The first result about strongly Sperner posets is the discovery by Erdős [8] in 1945 that B_n is a strongly Sperner poset.

A particularly interesting class consists of strongly Sperner posets which are *RSU*. These are now called *Peck posets* (in honor of the imaginary mathematician, G.W. Peck). The lattices B_n, $L_n(q)$, and $L(m, n)$ are all Peck posets.

The areas of linear algebra, category theory, and network flows have all provided tools to develop properties and applications of strongly Sperner posets and related families. Among the most interesting results are product and quotient theorems (see the surveys [7, 10, 21]). This theory played a surprising yet essential role in the solution of a long-standing number theory problem of Erdős and Moser by Stanley [20].

The strong Sperner property S is closely related to a condition on chains in the subposet induced on the largest ranks of P: We say P has the *chain property* T [12] if for all $k \geqslant 0$, there exists a set of M_k disjoint chains in P that each meet every rank set P_i with $|P_i| \geqslant M_k$. This property was first used by Stanley [20] for RSU posets as a way of deriving the Sperner property S_1. (It is called 'T' in honor of Stanley, since we already used 'S' for Sperner!) It is important to note that there may be no single set of chains having this property for all k, e.g., consider Example 15 in Fig. 4 [12]. This poset has property T, by taking the six middle diagonals for $k = 0, 1$ and the three long vertical chains for $k = 2, 3$, but no single set of chains works for all k. Here is the main result linking the Sperner property to conditions on chains.

Theorem 1 (Griggs [12]). *For graded posets, the strong Sperner property S implies the chain property T. For rank-unimodal graded posets, the chain property T implies the strong Sperner property S, but it is not true for graded posets in general.*

Example 12 in Fig. 4 shows a poset that has the chain property T, yet does not have the strong Sperner property S [12]. Indeed, this poset is not even a Sperner poset.

We noted above that in general, a graded poset P that has the chain property T will not contain a *single* collection of chains realizing T. This happens only for a partition of P into chains, say $\mathscr{C} = \{C_1, C_2, \ldots\}$, such that for all k, M_k chains in $\mathscr{C}$ meet every one of the $k + 1$ largest ranks in P. Then M_n chains meet every rank, and removing them from P deletes the smallest rank in P. Then $M_{n-1} - M_n$ chains meet every rank in what remains of P. Continuing in this fashion, we learn that the rank sets of the chains are *nested* in the sense that

$$r(C_i) \subseteq r(C_j) \quad \text{if } |C_i| \leqslant |C_j|.$$

When such a partition exists, we say P has the *nested chain property* N.

A chain partition $\mathscr{C} = \{C_1, C_2, \ldots\}$ of a finite poset P induces an upper bound on the size of any k-family $F \subseteq P$, since

$$|F| = \sum_i |F \cap C_i| \leqslant \sum_i \min(k, |C_i|). \tag{1}$$

When the right-hand side of (1) equals $d_k(P)$, $\mathscr{C}$ is said to be *k-saturated.* There is an impressive theory about k-saturated chain partitions of finite posets, started by Greene and Kleitman [10]. When a graded poset P has a nested chain partition $\mathscr{C}$, the right-hand side of (1) above is $M_0 + \cdots + M_{k-1}$, which is the general lower bound on $d_k(P)$, so that $\mathscr{C}$ is k-saturated for all k. (Then $\mathscr{C}$ is said to be *completely saturated.*) Moreover, since $d_k(P) = M_0 + \cdots + M_{k-1}$, P is a strongly Sperner poset.

Proposition 2 (Griggs [12]). *If the graded poset P has the nested chain property N, then it has the strong Sperner property S.*

The most famous class of nested chain posets is the *symmetric chain orders*, which are the *RSU* nested chain posets. They possess decompositions into chains that are skipless and symmetric about the middle rank, $r(P)/2$. Examples of these include the Boolean lattice B_n [3] and the linear lattice $L_n(q)$ [1, 11]. A long-standing open problem due to Stanley is to decide whether the lattice $L(m, n)$ is in general a symmetric chain order.

The converse of Proposition 2 fails, even for the class of *RSU* posets, as shown in Example 15.

We now introduce a new chain condition that will arise later in connection with our matching properties. We say a graded poset P has the *skipless Dilworth property D* if there exists a partition of P into $d_1(P)$ chains (the minimum possible, called a *Dilworth decomposition*) such that every chain is skipless. A variety of posets that possess or do not possess property D is given in Fig. 4.

4. Matching conditions

For a graded poset P with just two levels (i.e., $r(P) = 1$), the Sperner properties S_1 and S and the chain property T are all equivalent to the existence of a matching from the smaller to the larger of the two levels. We consider in this section what happens in general graded posets with the property that between every consecutive pair of rank sets, there is a matching from the smaller to the larger one, call this the *matching property* M_1.

Canfield [5] introduced this idea for the partition lattice, Π_n, determining in an asymptotic sense as $n \to \infty$ which consecutive pairs of levels have such matchings. For comparison with his earlier disproof of Rota's conjecture, notice that to prove that Π_n is not a Sperner poset, it suffices to prove that there is no such matching between some two levels, one of which has maximum size. It is known that the *RU* lattice Π_n has at most two ranks of maximum size. Use $\Pi_{n,k}$ to denote the set of partitions of $\{1, \ldots, n\}$ into k blocks. Then the least value $k = K_n$ that maximizes $|\Pi_{n,k}|$ satisfies $K_n \sim n/\log n$. Canfield builds on his earlier work and on work of Mullin, Shearer, and Kung to prove that there exist L_n and R_n such that

$$\Pi_{n,k} \hookrightarrow \Pi_{n,k+1} \text{ for } k < L_n \quad \text{and} \quad \Pi_{n,k} \hookrightarrow \Pi_{n,k-1} \text{ for } k > R_n,$$

and otherwise such matchings between consecutive levels do not exist. Furthermore, these thresholds satisfy

$$L_n \sim (\log 2) K_n \quad \text{and} \quad R_n \sim (\log 4) K_n \text{ as } n \to \infty.$$

We also introduce the *strong matching property M*, which means there is a matching between *each* pair of ranks, consecutive or not. Clearly, M implies M_1. One easily obtains such matchings M from the chains given by property T, when it holds.

Proposition 3. *A graded poset P with the chain property T has the strong matching property M.*

We are now ready to present our result relating the matching conditions to the other properties. It is the motivation for introducing the skipless Dilworth property D.

Theorem 4. *A graded poset P has both the Sperner property S_1 and the skipless Dilworth property D if and only if it is rank-unimodal and has the matching property M_1.*

Proof. Let P_j be a maximum-sized rank set in a graded poset P.

First, suppose P has properties S_1 and D. Let $\mathscr{C}$ be a partition of P into $|P_j| = M_0$ skipless chains. Then every chain meets P_j, and for each $i > j$ (resp., $i < j$), every chain that meets P_i also meets P_{i-1} (resp., P_{i+1}). It follows that P is RU. Further, $\mathscr{C}$ provides a matching between each pair of consecutive levels, so P has the matching property M_1.

Conversely, it is clear that if P is RU, then by linking together matchings between consecutive ranks using M_1, we obtain a partition into $|P_j|$ skipless chains, so that P has properties S_1 and D. $\square$

Corollary 5. *Let P be a graded poset with the skipless Dilworth property D. Then P has the Sperner property S_1 and the chain property T if and only if it is rank-unimodal and has the strong Sperner property S.*

Proof. When P has properties S_1 and D, then by the preceding theorem, it is RU. Then if P also has property T, it is a strongly Sperner poset by Theorem 1. Conversely, any strongly Sperner poset has properties S_1, by definition, and T, by Theorem 1. $\square$

5. Cutset conditions

We have not seen a discussion in the literature relating the Sperner properties and cutset sizes, although there is a natural relationship, which we shall describe next. Since any k nonempty ranks in a graded poset P of rank n form a k-cutset, we have

$$g_k \leqslant M_n + \cdots + M_{n-k+1} \quad (1 \leqslant k \leqslant n+1). \tag{2}$$

We say P has the *cutset property* C_1 if the smallest rank is a cutset of minimum size. If equality holds in (2) for every k, $1 \leqslant k \leqslant n+1$, we say P has the *strong cutset property* C.

The chain property T has a nice interpretation in terms of cutsets, via network flows [15]. One can represent poset P naturally by a network, representing each element $x \in P$ by a pair of nodes U_x, V_x and an arc (U_x, V_x) of unit capacity. Insert an infinite capacity arc (V_x, U_y) whenever y covers x in P. Finally, create a source node s and sink node t and insert infinite capacity arcs (s, U_x) for $x \in P_0$ and (V_y, t) for $y \in P_n$. Then integral network flows correspond to disjoint maximal chains in P. By network flow theory, the maximum number of disjoint maximal chains in P equals the minimum cutset size, g_1. To characterize property T, we now apply this idea for any k to the subposet consisting of the k largest ranks in P. (To check for property T, it suffices to do this for values k such that $M_{k-1} > M_k$.)

Theorem 6. *A graded poset P has the chain property T if and only if for all k, the subposet consisting of the k largest ranks in P has the cutset property C_1.*

There is actually a very simple connection between the cutset and Sperner properties. Since all maximal chains have the same size, the k-cutsets are precisely the complements in P of the $(n + 1 + k)$-families. Thus, we have:

Proposition 7. *A graded poset P has the strong Sperner property S if and only if it has the strong cutset property C.*

6. LYM posets

For comparison with the properties above, we survey work on another special class, the *LYM* posets. Consider a pair of adjacent levels A and B in a graded poset P, with $|A| \leqslant |B|$. We can characterize when $A \hookrightarrow B$ in terms of the shadows $\Gamma(X)$ for $X \subseteq A$. By Hall's theorem, $A \hookrightarrow B$ if and only if for all X, $|\Gamma(X)| \geqslant |X|$. A stronger condition we can impose is to demand that the proportion of elements from B in the shadow be larger than the proportion of elements from A in X:

$$\frac{|\Gamma(X)|}{|B|} \geqslant \frac{|X|}{|A|} \quad \text{for all } X \subseteq A. \tag{3}$$

When (3) holds for every pair of consecutive ranks in the graded poset P, we obtain the *normalized matching property NM* of Graham and Harper [9]. Note that the roles of A and B can be interchanged in checking (3), i.e., it does not matter whether or not $|A| \leqslant |B|$. So to prove that P has the *NM* property, it suffices to check that (3) holds when $A = P_k$ and $B = P_{k+1}$ for $0 \leqslant k \leqslant r(P) - 1$.

Kleitman [16] found that a very different property is equivalent to the *NM* property. He extended to general ranked posets an inequality that strengthens Sperner's theorem for B_n. Named the *LYM property*, after some of its discoverers (who include Lubell, Yamamoto, Meshalkin, and others, especially Bollobás), it requires that

$$\sum_i \frac{|A \cap P_i|}{|P_i|} \leqslant 1 \quad \text{for every antichain } A \subseteq P. \tag{4}$$

This LYM inequality (4) easily implies the strong Sperner property S. Thus, LYM posets also have the chain property T and the matching property M, by Theorem 1 and Proposition 3.

Of course, property M can be deduced directly from the NM property using Hall's theorem. There is another proof using cutsets that LYM posets P have property T. Clearly, the subposet induced on any union of rank sets of an LYM poset P is also LYM by (4). By Theorem 6, to prove P has T, it suffices to prove that every LYM poset has the cutset property C_1. To do this, consider any subset F of an LYM poset P with $|F| < |M_n|$. We will show some maximal chain avoids F. Note that $\sum_i (|F \cap P_i|/|P_i|) < 1$. The proportion of elements of the top rank that avoids F is $1 - (|F \cap P_n|/|P_n|)$. The shadow of these elements going down has proportion at least as large in P_{n-1}. From this, we must subtract at most the proportion of elements belonging to F, which is $|F \cap P_{n-1}|/|P_{n-1}|$. Continuing on down, we find at the bottom that a nonzero proportion of elements survives in successive shadows in $P - F$, allowing us to construct a maximal chain avoiding F.

It remains to resolve how LYM posets relate to the nested chain property N. The seven-element poset $P + Q$, where $P = C_3$ and $Q = B_2 = A_1 \oplus A_2 \oplus A_1$ has property N but is not LYM. This example is also RSU. On the other hand, no example is known of an LYM poset that does not have property N. Anderson and Griggs independently settled the case for RSU posets.

Theorem 8 (Anderson [1] and Griggs [11]). *If P is an LYM poset that is rank-symmetric and rank-unimodal, then it has the nested chain property N, i.e., is a symmetric chain order.*

This was used to prove, for example, that the linear lattice $L_n(q)$ is a symmetric chain order. Griggs [11] conjectures that all LYM posets possess property N, but there has been no real progress on this problem, not even for RU posets that are regular in the sense that for all i, each element of P_i covers the same number of elements and each is covered by the same number of elements.

7. Overview and examples

We present Venn diagrams in Figs. 1–3 to summarize the possible implications relating the various properties in the paper. Each oval is labelled by the property it represents, so that if property A holds whenever property B holds, then oval A contains oval B. In this case, we write $B \subseteq A$. Each nonempty region corresponds to a collection of properties that actually occurs. A list of all properties is at the end of the section. We omit the properties $LYM = NM$ from the figures to keep them simpler (and because whether $LYM \subseteq N$ in general is unresolved).

First consider general graded posets. The situation is rather complicated, so in Fig. 1 we show the implications disregarding the skipless Dilworth property D. These

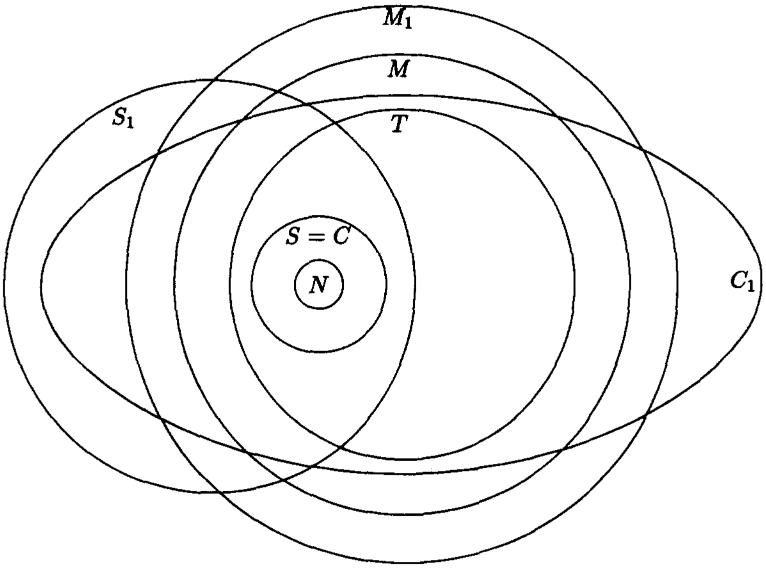


Fig. 1. Property implications for general graded posets, disregarding property D.

are summarized as follows, where $\subset$ denotes proper inclusion:

$$\begin{Bmatrix} \emptyset \neq N \subset S = C \subset S_1 \cap T \subset S_1,\ T \\ T \subset M \subset M_1 \\ T \subset C_1 \end{Bmatrix}. \tag{5}$$

All nonempty regions in Fig. 1 are realized by graded posets not satisfying D from Fig. 4 or by an easy modification. Ordinal sums with antichains are handy for destroying D or adding C_1. By design, none of Examples 1–11 has the cutset property C_1. It is easily checked that for any Sperner, chain, or matching property, as well as for the strong cutset property C, that a graded poset P has the property if and only if $A_1 \oplus P$ has the property. But $A_1 \oplus P$ has the cutset property C_1, independent of P. For example, $A_1 \oplus$ Example 9 has properties S_1, M_1, C_1.

For general graded posets with property D, a few possibilities vanish, and we obtain Fig. 2, which shows implications (5) together with

$$\begin{Bmatrix} S_1 \subset M_1 \\ S_1 \cap T = S \end{Bmatrix}. \tag{6}$$

We remind the reader that we also know that in general graded posets,

$$\begin{Bmatrix} LYM = NM \subset S \\ N \not\subseteq LYM \end{Bmatrix}. \tag{7}$$

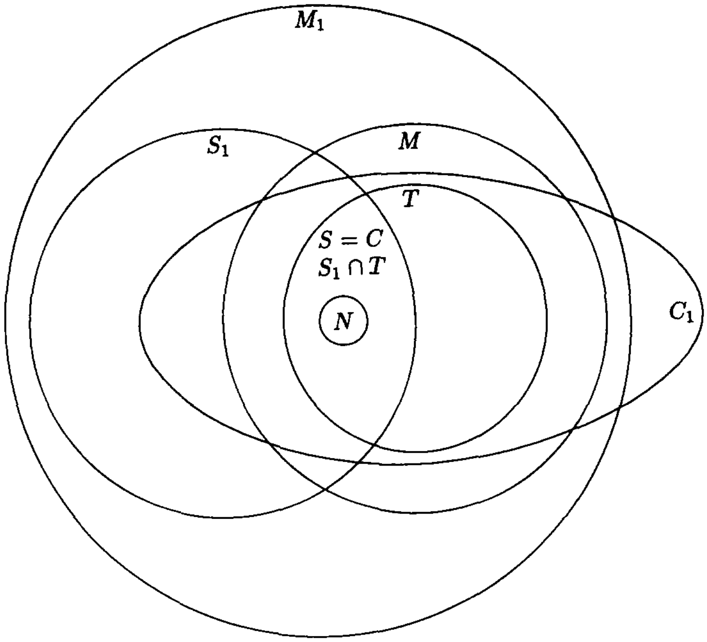


Fig. 2. Property implications for graded posets with property D.

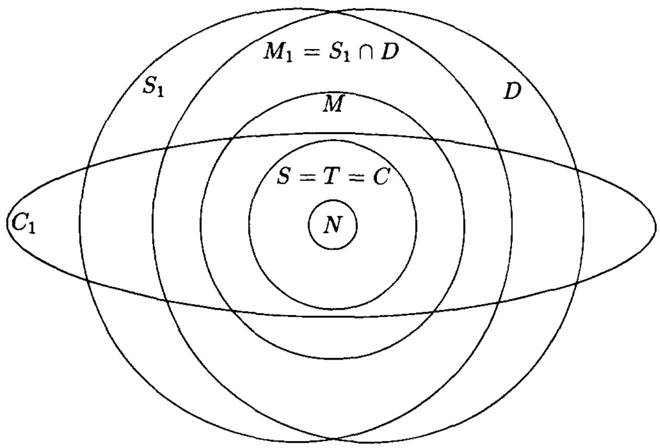


Fig. 3. Property implications for rank-unimodal graded posets.

What if we constrain the sequence of Whitney numbers?. If P is RU, we have the implications in Fig. 3:

$$\left\{\begin{matrix} \emptyset \neq N \subset S = T = C \subset M_1 = S_1 \cap D \subset S_1,\ D \\ C \subset C_1 \end{matrix}\right\}. \tag{8}$$

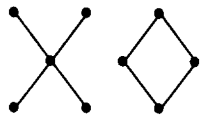
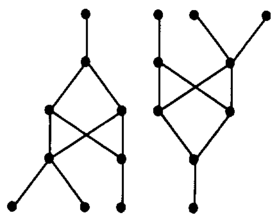
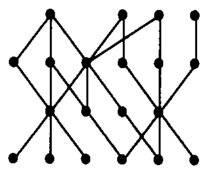
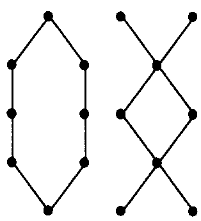
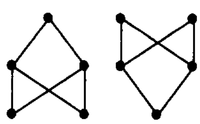
1.		RSU
2.		D, RSU
3.		D, M_1
4.	$A_5 \oplus A_4 \oplus Ex.3$	M_1
5.		D, M_1, M
6.	$A_5 \oplus A_4 \oplus Ex.5$	M_1,M
7.		RSU,S_1
8.		D,M_1,RSU,S_1

Fig. 4. List of examples and their properties.

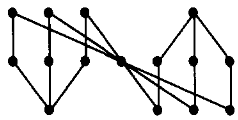
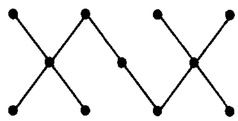
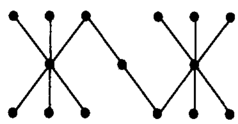
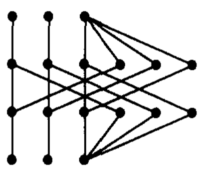
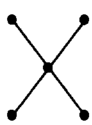
9.	$A_4 \oplus A_3 \oplus Ex.8$	M_1,S_1
10.		D, M_1, M, RSU, S_1
11.	$A_5 \oplus A_4 \oplus Ex.10$	M_1,M,S_1
12.		C_1,D, M_1, M, T
13.		C_1,M_1, M, T
14.	$A_7 \oplus Ex.13$	C_1,M_1,M,S_1,T
15.		C_1,C,D, M_1, M, RSU,S_1,S,T
16.	$A_2 \oplus A_1 \oplus Ex.15$	C_1,C,M_1,M,S_1,S,T
17.	A_1	All properties
18.		C_1,C,LYM,M_1,M,N,NM,S_1,S,T

Fig. 4. Continued.

In fact, all possible combination of properties in Fig. 3 arise with RSU posets, which shows that no further implications than (8) hold for those properties. By design, the RU examples in Fig. 4 are all RSU. To add property C_1 to an RSU poset P, the poset $A_1 \oplus P \oplus A_1$ is also RSU and retains all of the other properties, except that it now has the cutset property. For RU posets, we still have (7), but for RSU posets we add $LYM \subset N$.

We conclude by discussing the broader class of ranked posets. The properties we defined make sense for all ranked posets, with the exception of the cutset properties. For example, taking $P = C_3 + C_2$ with both minimal elements in rank zero, we have $g_1 = 2 > M_2 = 1$, which violates inequality (2). The problem is that not every rank set is a cutset. Disregarding the cutset properties, it can be checked that all results for the other Sperner, chain, and matching properties extend to this larger class.

Appendix: List of properties of graded posets

C_1	cutset property
C	strong cutset property
D	skipless Dilworth property
LYM	LYM property
M_1	matching property
M	strong matching property
N	nested chain property
NM	normalized matching property
RSU	rank-symmetric and rank-unimodal
RU	rank-unimodal
S_1	Sperner property
S	strong Sperner property
T	chain property

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